The Price of Anarchy in Auctions
Part I: Introduction and Motivation

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Single-item Auction Problem:

Given:

- one item for sale.
- \( n \) bidders (with unknown private values for item, \( v_1, \ldots, v_n \))
- Bidders’ objective: maximize utility = value \(-\) price paid.

Design:

- Auction to solicit bids and choose winner and payments.
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Possible Auction Objectives:

- Maximize social welfare, i.e., the value of the winner.
- Maximize seller revenue, i.e., the payment of the winner.
The First-price Auction

1. Solicit sealed bids.
2. Winner is highest bidder.
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Question: How should you bid?
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Probability Density Function: $f(z) = \frac{1}{dz} \text{Pr}[v \leq z] = 1$. 
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Expectation: $E[v] = \int_0^\infty v f(v) \, dv = \int_0^\infty (1 - F(v)) \, dv$
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Expectation: $E[v] = \int_0^\infty v f(v) \, dv = \int_0^\infty (1 - F(v)) \, dv = 1/2$
First-price Auction: Symmetric Dists

Example: two bidders (you and me), uniform values.
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Conclusion 1: bidding “half of value” is equilibrium.
Conclusion 2: bidder with highest value wins
Conclusion 3: first-price auction maximizes social welfare!
Questions?
Bayes-Nash equilibrium (BNE)

**Def:** a *strategy* maps value to bid, i.e., $b_i(v_i)$. 
Bayes-Nash equilibrium (BNE)

Def: a strategy maps value to bid, i.e., \( b_i(v_i) \).

Def: the common prior assumption: bidders’ values are drawn from a known distribution, i.e., \( v_i \sim F_i \).
**Bayes-Nash equilibrium (BNE)**

**Def:** a *strategy* maps value to bid, i.e., $b_i(v_i)$.

**Def:** the *common prior assumption*: bidders’ values are drawn from a known distribution, i.e., $v_i \sim F_i$.

**Definition:** a *strategy profile* is in *Bayes-Nash Equilibrium (BNE)* if for all $i$, $b_i(v_i)$ is best response when others play $b_j(v_j)$ and $v_j \sim F_j$. 
Example: two bidders, $v_1 \sim U[0, 1]$, $v_2 \sim U[0, 2]$
First-price Auction: Asymmetric

**Example:** two bidders, $v_1 \sim U[0, 1]$, $v_2 \sim U[0, 2]$  

- $b_1(v) = \frac{2}{3v} (2 - \sqrt{4 - 3v^2})$  
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- highest-valued agent may not win in BNE \( \Rightarrow \) PoA > 1.

Asymmetric Equilibrium Solutions:
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- one uniform bidder, one constant bidder [Vickrey '61]
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- $U[\alpha_1, \beta_1], U[\alpha_2, \beta_2]$. [Kaplan, Samier '12]
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**Notes:** solved by differential equation, 50 years to solve general uniform case, only for two bidders.
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1. solve for equilibrium.

2. interpret quality of equilibrium. (e.g., for welfare or revenue)
Classic Analysis vs Price of Anarchy

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   - bidder strategies not obvious.
   - challenge: asymmetric distributions.
   - challenge: generalizations of single-item auctions.
   - challenge: other auctions run at same time.

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PoA Analysis: quantify performance without solving for equilibrium.
Questions?
**Thm:** for all distributions and BNE the first-price auction satisfies

$$\mathbb{E}[\text{BNE welfare}] \geq \frac{1}{2} \mathbb{E}[\text{OPT welfare}]$$
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**Proof Outline:**
1. Decompose \( \mathbb{E}[\text{BNE welfare}] = \mathbb{E}[\text{BNE utilities}] + \mathbb{E}[\text{BNE revenue}] \).
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   \[ \Rightarrow \mathbb{E}[\text{bidder’s BNE utility}] \geq \mathbb{E}[\text{utility from deviation}] \]
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3. *deviation covering lemma:* if bidder \( i \) deviates to \( b_i' = v_i/2 \)
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3. deviation covering lemma: if bidder \( i \) deviates to \( b'_i = v_i / 2 \)
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5. Sum over bidders, expectation over values:
   \[ \Rightarrow E[BNE \text{ utils}] + E[BNE \text{ revenue}] \geq \frac{1}{2} E[OPT \text{ welfare}] \]
Deviation Covering Lemma: \( u_i(v_i, v_i/2) + \mathbb{E}[\text{BNE revenue}] \geq \frac{1}{2} v_i \)

Proof by Picture:
Deviation Covering Lemma

Deviation Covering Lemma: \( u_i(v_i, v_i/2) + \mathbb{E}[\text{BNE revenue}] \geq \frac{1}{2}v_i \)

\begin{itemize}
  \item from bidder \( i \) (w. value \( v_i \))
    \begin{align*}
      b'_i &= v_i/2 = \text{deviation bid} \\
      u'_i &= u_i(v_i, b'_i)
    \end{align*}
\end{itemize}

Proof by Picture:
Deviation Covering Lemma:

**Deviation Covering Lemma:** \( u_i(v_i, v_i/2) + E[BNE revenue] \geq \frac{1}{2} v_i \)

**Proof by Picture:**

- \( b_i' = v_i/2 \) = deviation bid
- \( u_i' = u_i(v_i, b_i') \) = \( (v_i - b_i') \) Pr[bid \( b_i' \) wins].
Deviation Covering Lemma: \( u_i(v_i, v_i/2) + \mathbb{E}[\text{BNE revenue}] \geq \frac{1}{2} v_i \)

from bidder \( i \) (w. value \( v_i \))

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from auction (and other bids)

- \( G_i = \text{high competing bid dist.} \)

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Deviation Covering Lemma: \( u_i(v_i, v_i/2) + E[BNE \text{ revenue}] \geq \frac{1}{2}v_i \)

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**From auction (and other bids)**

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**Proof by Picture:**

![Diagram illustrating the Deviation Covering Lemma](image-url)
Deviation Covering Lemma: \( u_i(v_i, v_i/2) + \mathbb{E}[\text{BNE revenue}] \geq \frac{1}{2} v_i \)

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**from auction (and other bids)**

\( G_i = \text{high competing bid dist.} \)
\[ \mathbb{E}[\text{BNE revenue}] \geq \mathbb{E}[\text{competing bid}] \]

**Proof by Picture:**

[Diagram showing the relationship between bids and utilities, with \( G_i(b_i') \), \( u_i' \), and \( b_i' \) highlighted.]
Deviation Covering Lemma: \( u_i(v_i, v_i/2) + \mathbb{E}[\text{BNE revenue}] \geq \frac{1}{2} v_i \)

**Proof by Picture:**

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- From auction (and other bids)
  \[\mathbb{E}[\text{competing bid}] \geq \mathbb{E}[\text{competing bid}] = \int_0^\infty 1 - G_i(b) \, db\]

- Diagram showing the relationship between expected revenue and bid behavior.
Deviation Covering Lemma: \( u_i(v_i, v_i/2) + \mathbb{E}[\text{BNE revenue}] \geq \frac{1}{2} v_i \)

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**from auction (and other bids)**

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G_i = \text{high competing bid dist.} \\
\mathbb{E}[\text{BNE revenue}] \geq \mathbb{E}[\text{competing bid}] \\
= \int_0^\infty 1 - G_i(b) \, db
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**Proof by Picture:**

\[
\begin{align*}
    \mathbb{E}[\text{comp. bid}] &\leq G_i(b'_i) \\
    u'_i &\geq 0
\end{align*}
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Deviation Covering Lemma

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\[
G_i = \text{high competing bid dist.} \\
\mathbb{E}[\text{BNE revenue}] \geq \mathbb{E}[\text{competing bid}] \\
= \int_0^\infty 1 - G_i(b) \, db
\]

**Proof by Picture:**

\[
\begin{align*}
    \mathbb{E}[\text{comp. bid}] &= u'_i \geq \int_0^b G_i(b) \, db \\
    &= \frac{1}{2} \times v_i
\end{align*}
\]
Questions?
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This tutorial: PoA for auctions (as games of incomplete information)
Overview of Tutorial

Part I: Introduction and motivation.

Part II: Smoothness Framework
(extension theorems, correlated dists., auction composition)

... coffee break ...

Part III: Standard Examples
(position auctions, multi-unit auctions, matching markets, combinatorial auctions)

Part IV: BNE Characterization and Consequences
(BNE characterization, symmetric BNE, solving, uniqueness, revenue)
Questions?