The Price of Anarchy in Auctions

Part IV: BNE Characterization and Consequences

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Part IV: Overview

Motivating Question: Can BNE characterization help PoA analyses?

Part IV: Overview

- BNE characterization.
- Solving for symmetric BNE.
- Uniqueness of symmetric BNE ($\Rightarrow$ PoA $= 1$).
- Revenue Optimization
- PoA for Revenue

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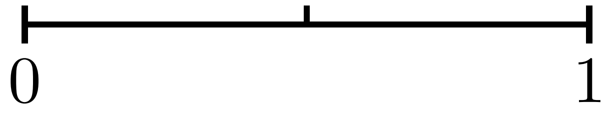

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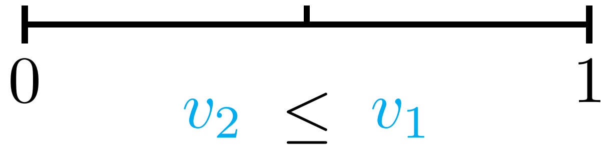

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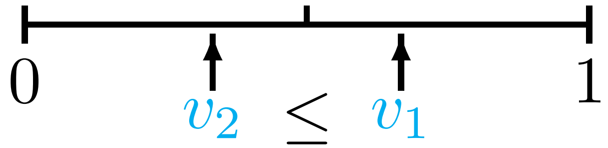

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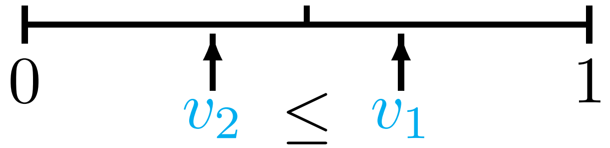

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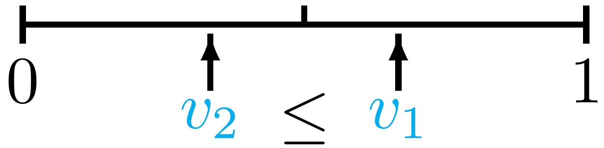

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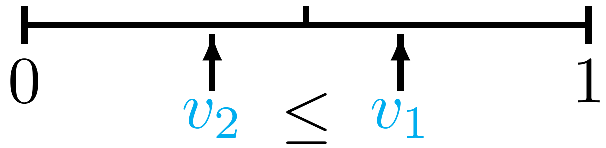

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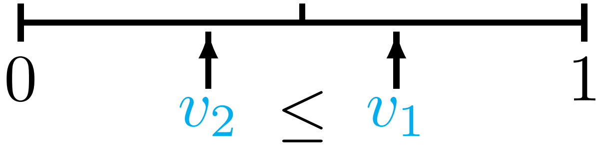

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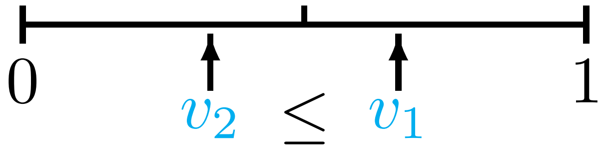

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Surprising Result: first-price and second-price auctions have same expected revenue in BNE.

BNE Characterization

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Analogously, define $\mathbf{p}$, $\mathbf{p}(\mathbf{v})$, and $p_i(v_i)$ for payments.

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Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
2. *payment identity (PI)*: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$.

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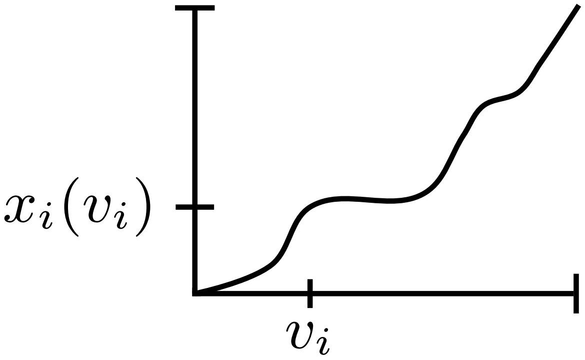

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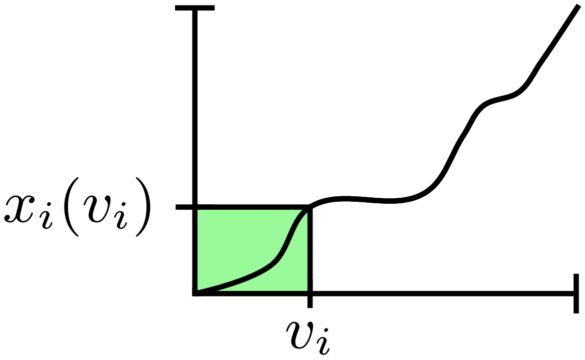

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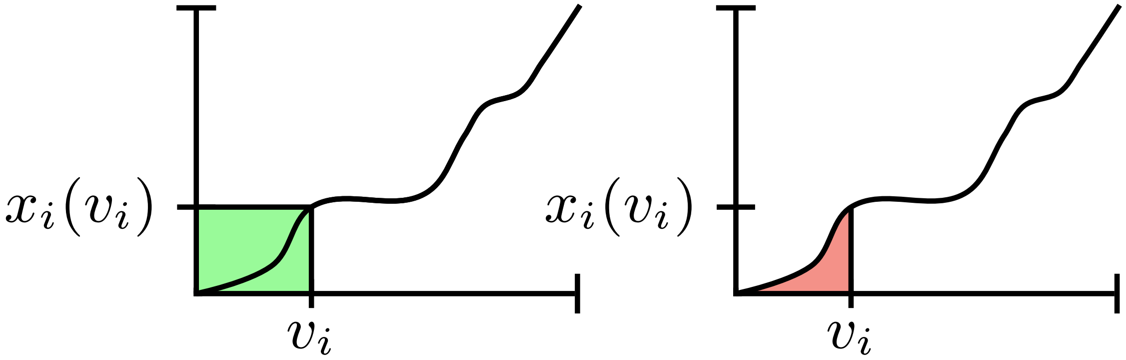

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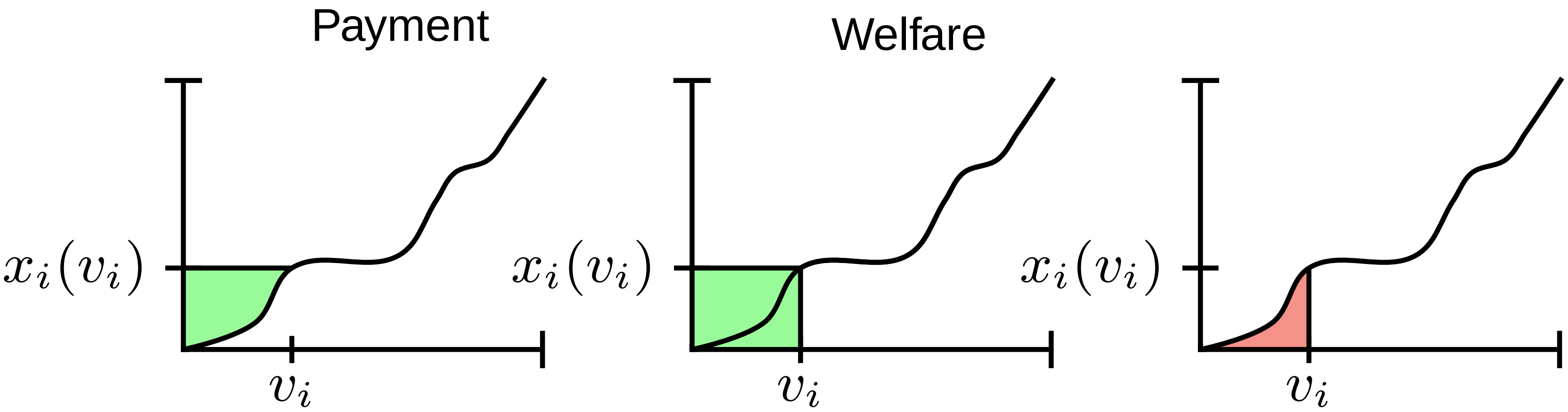

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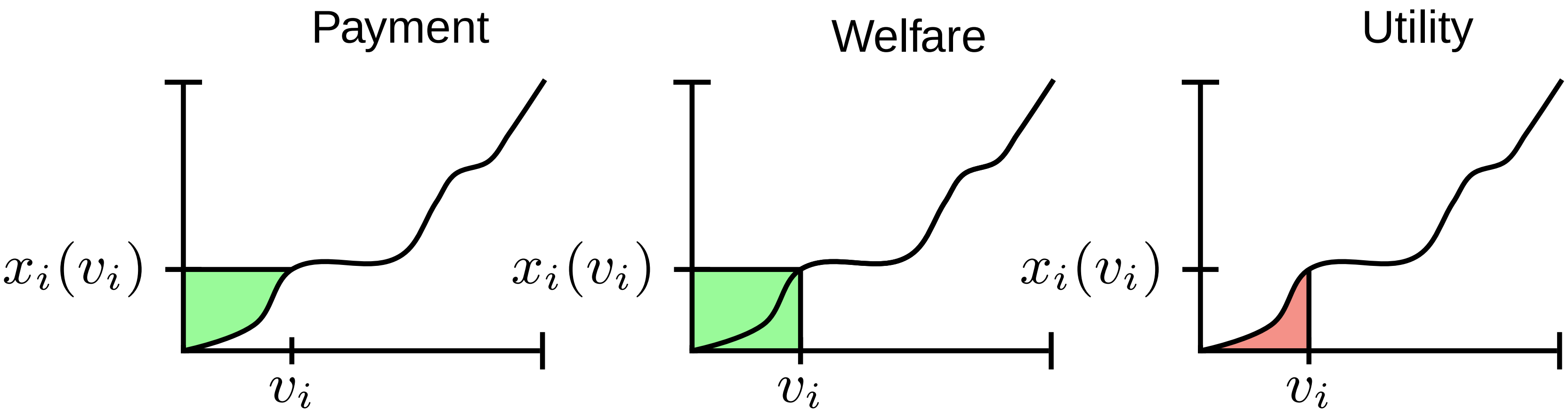

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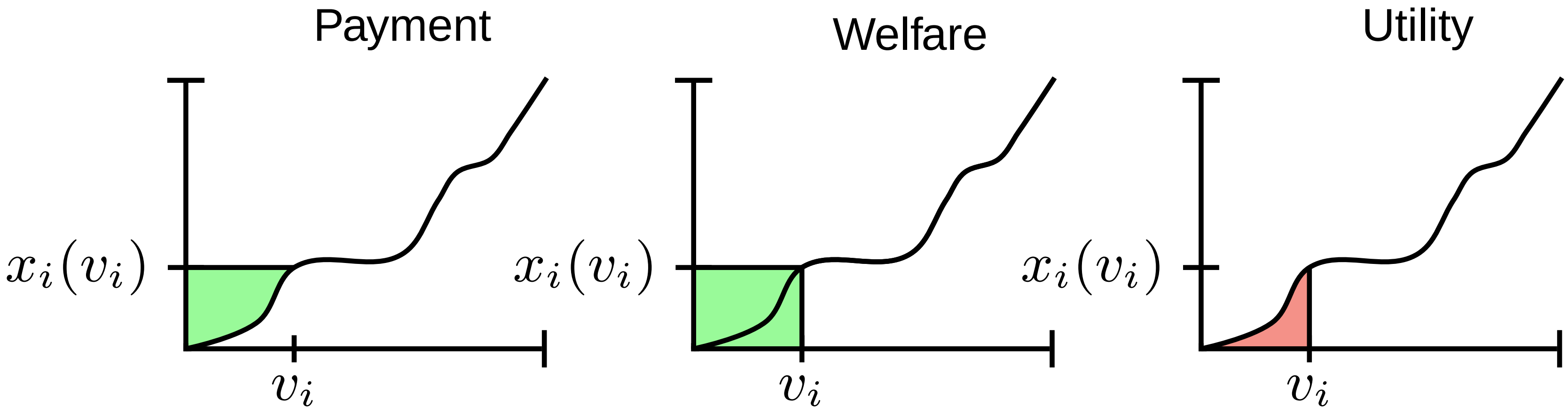

Consequence: *(revenue equivalence)* in BNE, auctions with same outcome have same revenue (e.g., first- and second-price auctions)

Example: Solving for BNE (Symmetric)

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3. Verify guess and BNE: $b(v)$ strictly increasing, symmetric

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Questions?

First-price Auction PoA $= 1$

(for symmetric distributions)

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Outline: Uniqueness of BNE (for i.i.d., continuous distributions)

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Conclusion: first-price auction PoA = 1 for symmetric distributions.
(generalizes to order-based auctions, e.g., position auctions)

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For general strategies:

- If one bidder has pointmass in bid distribution, other bidders must skip this bid ($\Rightarrow$ asymmetric).

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1. continuous
2. strictly increasing

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Corollary: n-player first-price auctions have no asymmetric equilibria. (iid, continuous, bounded dist.)

Proof of Corollary:

- player 1 & 2 face random reserve "$\max(b_3, \ldots, b_n)$"
- by theorem, their bid function is symmetric.
- same for player 1 and i.
- so all bid functions are symmetric.

Main Theorem

Two formulas for an agent's utility:

(1) $u(v) = (v - b(v)) \cdot x(v)$. (first-price payment rule)

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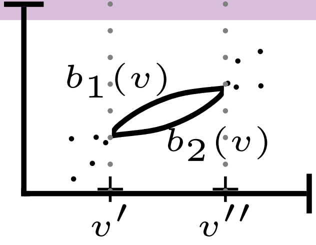

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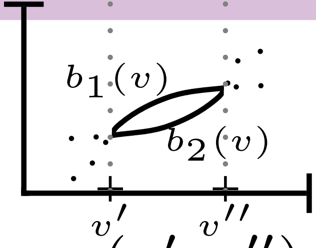

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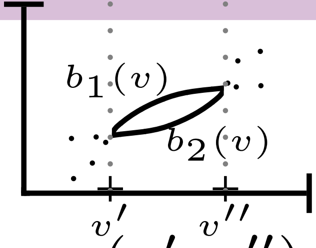

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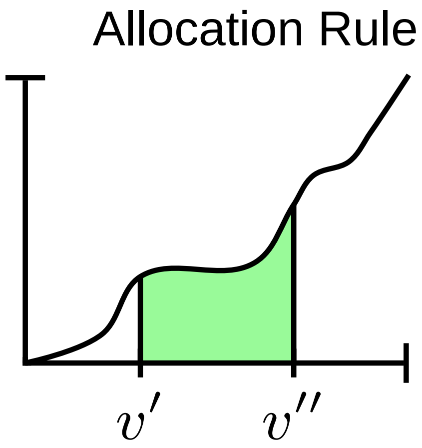
- so by (2): $u_1(v'') - u_1(v') = \int_{v'}^{v''} x_1(z) dz$



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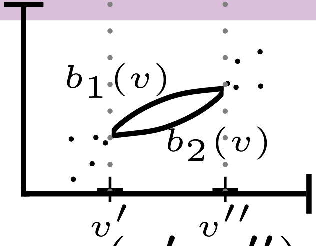

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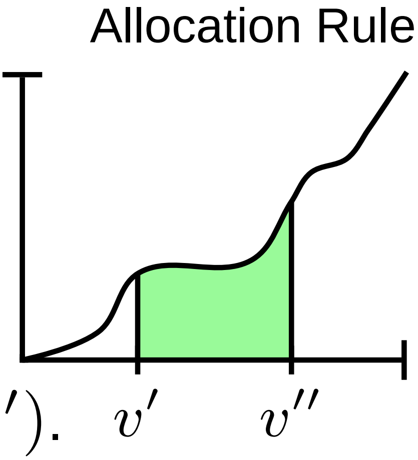


Claim 1: at v if $b_1(v) > b_2(v)$ then $x_1(v) > x_2(v)$ (and equal if equal)

Main Theorem

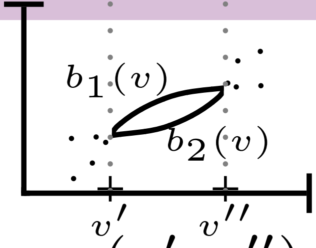
Two formulas for an agent's utility:

(1) $u(v) = (v - b(v)) \cdot x(v)$. (first-price payment rule)

(2) $u(v) = \int_0^v x(z)\,dz$. (payment identity / revenue equivalence)

Thm: 2-player, i.i.d., continuous, bounded first-price auctions with a random (unknown) reserve have no asymmetric equilibrium.

Proof: (by contradiction)



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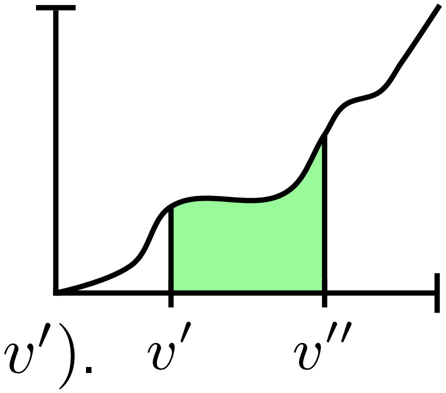
- but by Claim 1 and (1): $u_1(v') = u_2(v')$ and $u_1(v'') = u_2(v'')$

Allocation Rule



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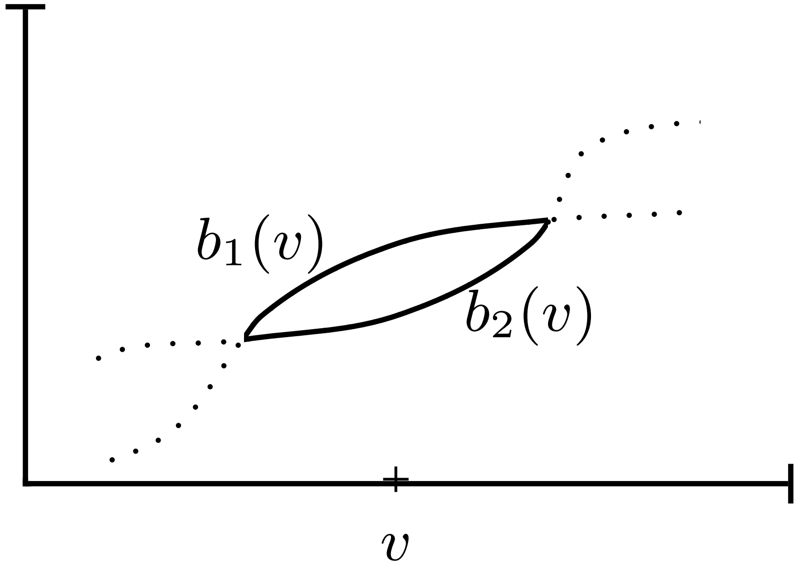
Bid Functions



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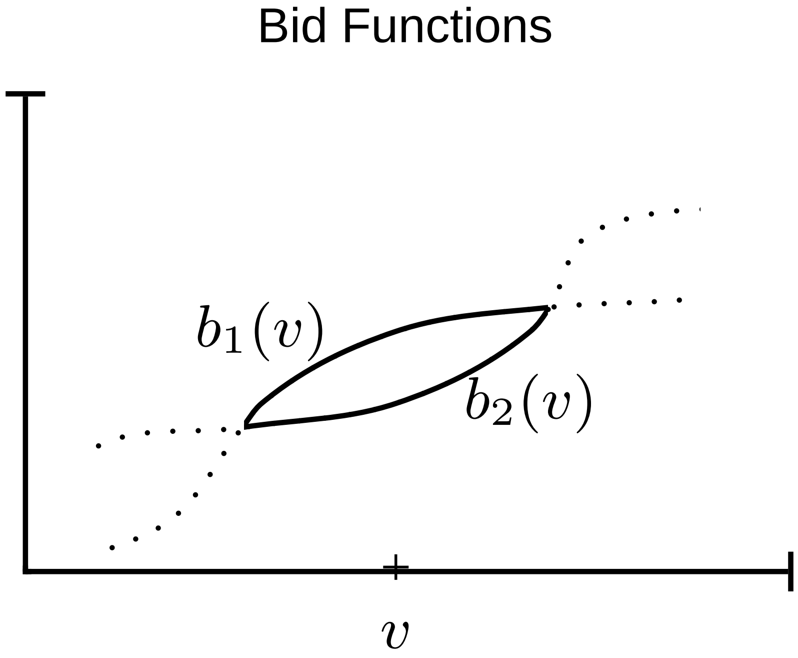

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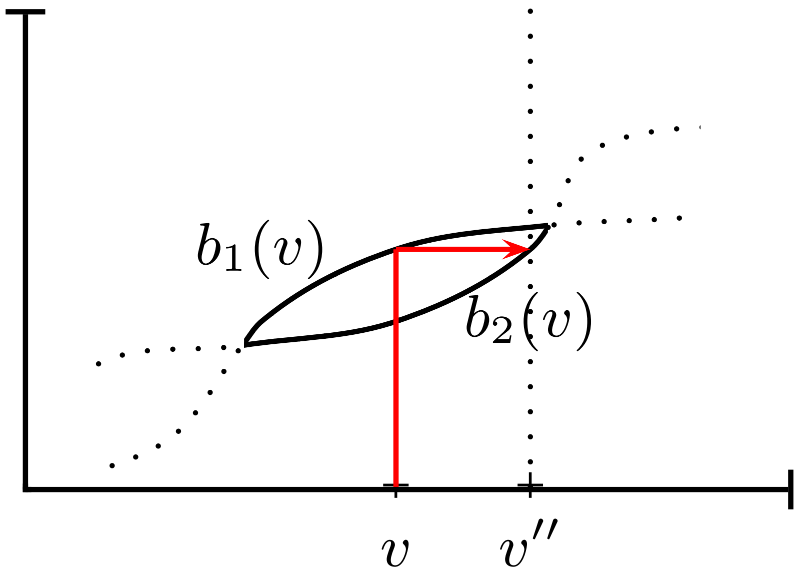

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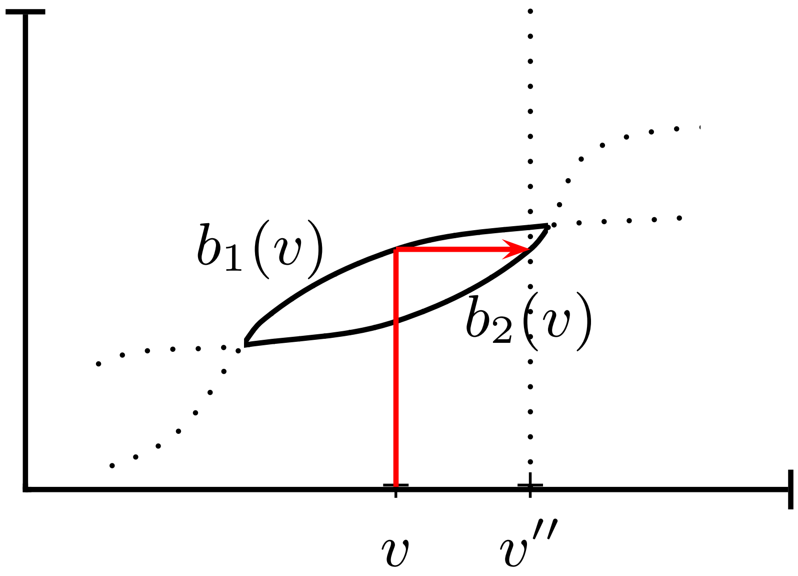

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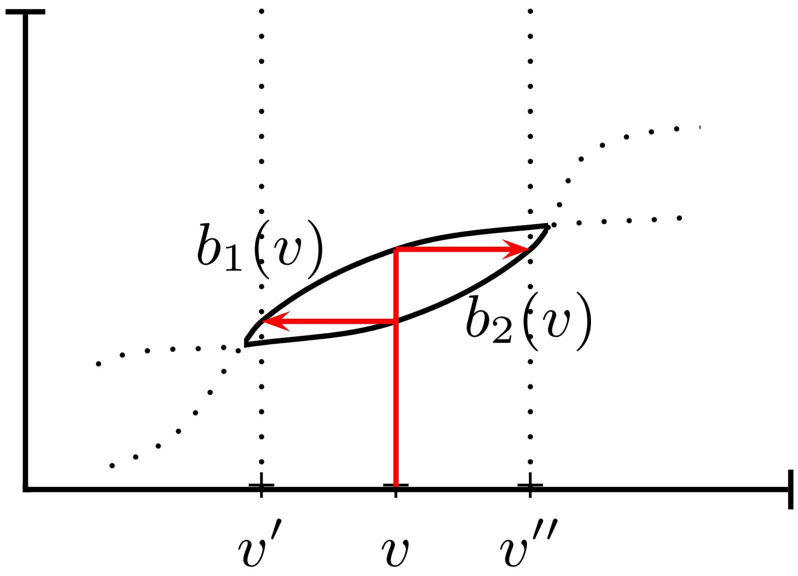

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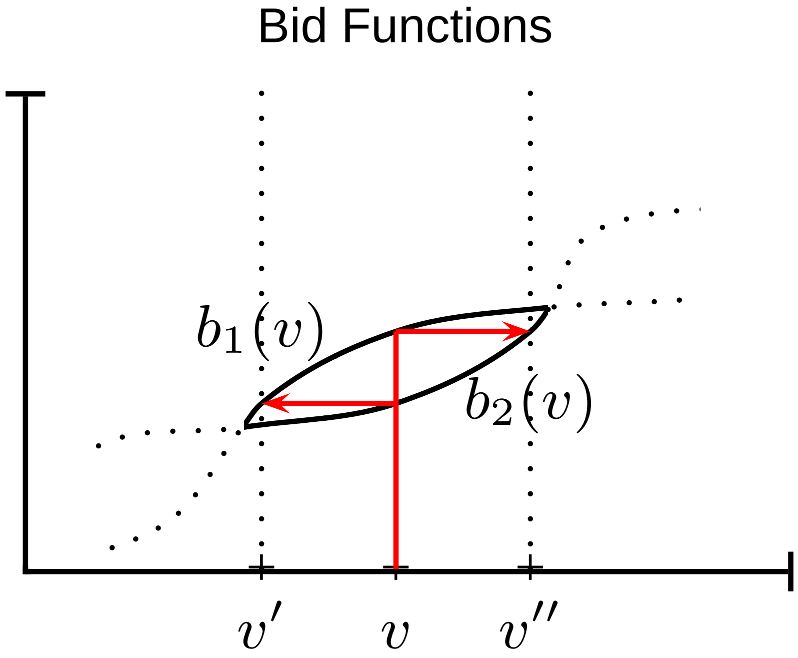

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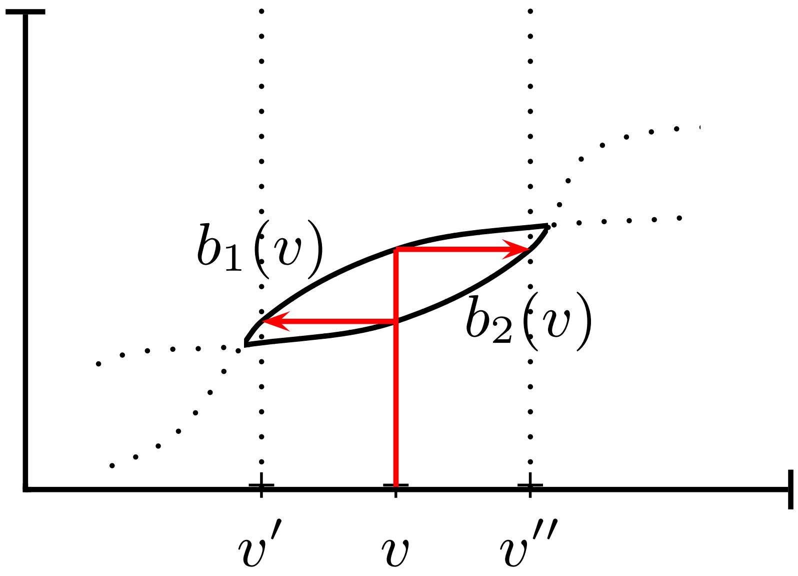
Bid Functions



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Questions?

Revenue-optimal Auctions

Optimizing BNE

Def: *virtual value* for i is $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$.

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- optimize revenue without incentive constraints (i.e., monotonicity).

  $\Rightarrow$ winner is agent with highest positive virtual value.

- check to see if incentive constraints are satisfied.

  $\Rightarrow$ if $\phi_i(\cdot)$ is monotone then mechanism is monotone.

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Proof: expected virtual valuation of winner $=$ expected payment.

Proof of Lemma

Recall Lemma: In BNE, $\mathbf{E}[p_i(v_i)] = \mathbf{E}\left[\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right) x_i(v_i)\right]$.

Proof Sketch:

- Use characterization: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(v)dv$.
- Use definition of expectation (integrate payment × density).
- Swap order of integration.
- Simplify.

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What is optimal single-item auction for $U[0, 1]$?

Optimal Auction for $U[0, 1]$

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- $F(v_i) = v_i$.
- $f(v_i) = 1$.
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- So, optimal auction is Second-price Auction with reserve 1/2!

Questions?

PoA for Revenue

Second-price (DSE) versus First-price (BNE)

Second-price auction (in dominant strategy equilibrium):

	symmetric	asymmetric
social surplus	1 [Vickrey ’61]	1 [Vickrey ’61]
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5. Sum over bidders, expectation over values:
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Covering Lemmas

Value Covering Lemma: $u_i(v_i) + \mathbf{E}[i\text{'s comp. bid}] \geq \frac{e-1}{e} v_i$

Revenue Covering Lemma: $\forall \mathbf{x}'$, $\mathbf{E}[\text{rev.}] \geq \sum_i \mathbf{E}[i\text{'s comp. bid}]\, x'_i$

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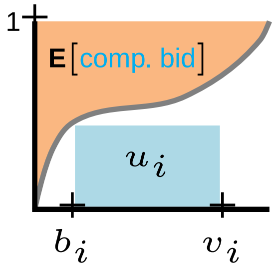
from bidder i (w. value v_i)

$$b_i = b_i(v_i) = \text{BNE bid}$$
$$u_i = u_i(v_i) = \text{BNE utility}$$
$$= (v_i - b_i)\ \underbrace{\mathbf{Pr}[\text{bid } b_i' \text{ wins}]}_{\mathbf{Pr}[\text{competing bid} \leq b_i'] = G_i(b_i')}.$$

from auction (and other bids)

$G_i =$ competing bid dist.

$$\mathbf{E}[\text{BNE revenue}]$$
$$\geq \mathbf{E}[\text{competing bid}]$$
$$= \int_0^\infty 1 - G_i(t)\, dt$$



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Value Covering Lemma: $u_i(v_i) + \mathbf{E}[i\text{'s comp. bid}] \geq \frac{e-1}{e} v_i$

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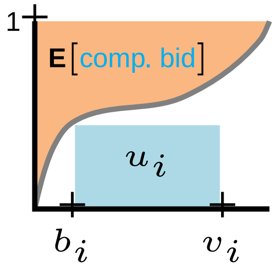

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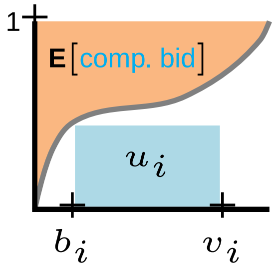

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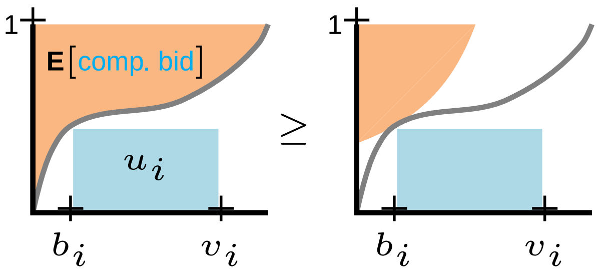

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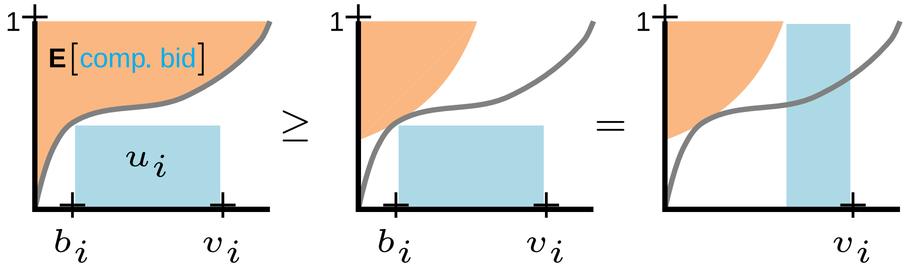

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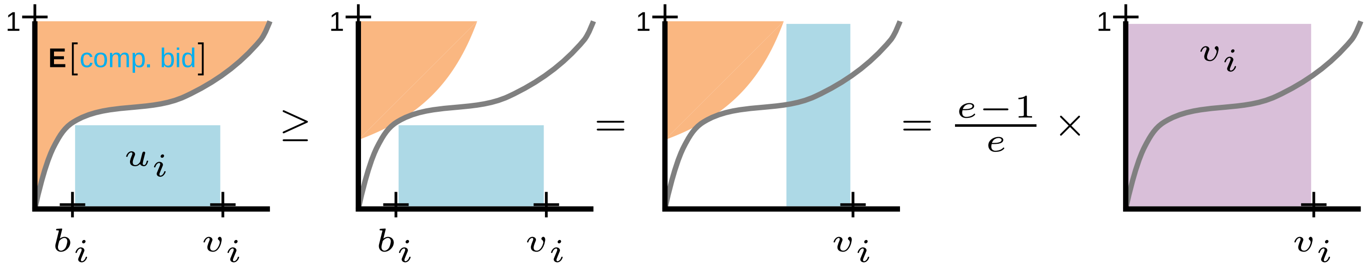

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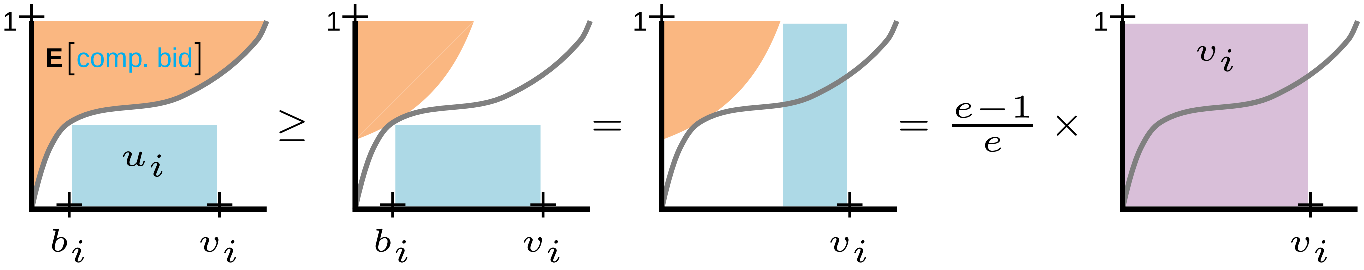

Conclusion: first-price auction has PoA $\frac{e}{e-1} \approx 1.58$.

Reserves and Covering Lemmas

Goal: first-price auction with monopoly reserves has good PoA.

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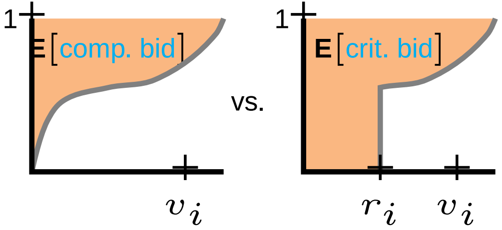

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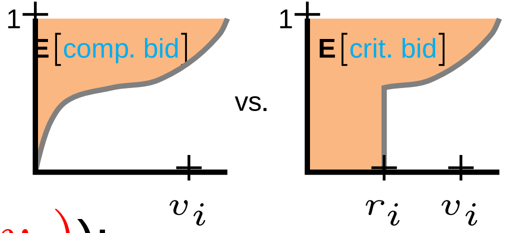

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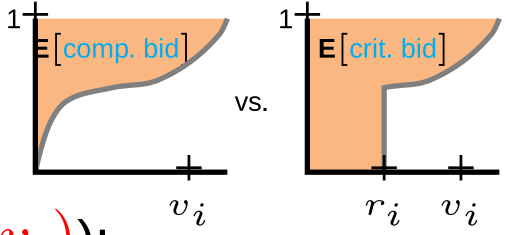

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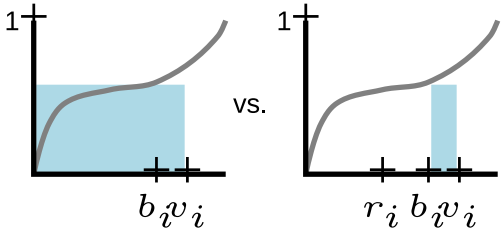

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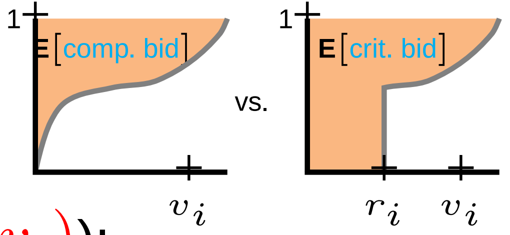

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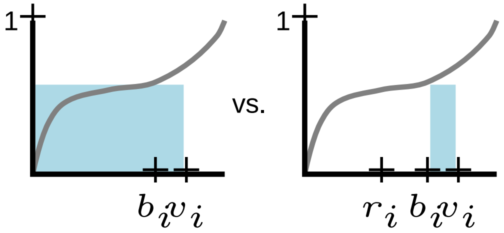

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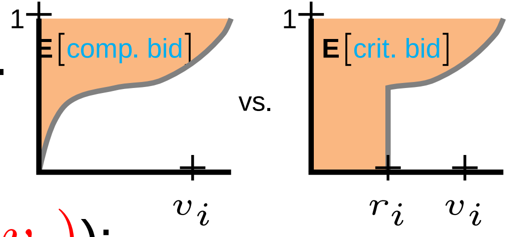

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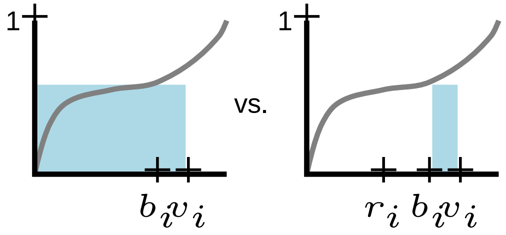

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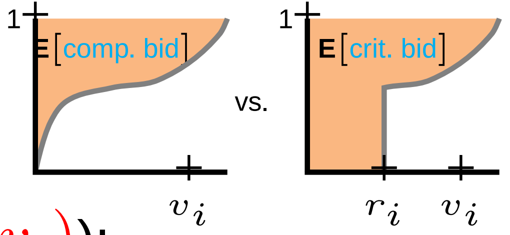

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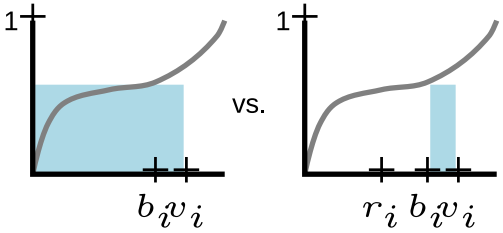

- Sum over bidders, expectation over values:

  $\Rightarrow \quad \mathbf{E}[\text{BNE welfare}] + \mu\, \mathbf{E}[\text{BNE revenue}] \geq \lambda\, \mathbf{E}[\text{OPT welfare}]$

Conclusion: PoA $= \frac{1+\mu}{\lambda}$ (versus optimal welfare with reserves)

Revenue Extension

Recall Proof Sketch: for welfare

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Thm: first-price auction w. monopoly reserves has PoA $\leq \frac{2e}{e-1} \approx 3.16.$

Questions?

Overview of Tutorial

Part I: Introduction and motivation.

Part II: Smoothness Framework
(extension theorems, correlated dists., auction composition)

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Part III: Standard Examples
(position auctions, multi-unit auctions, matching markets, combinatorial auctions)

Part IV: BNE Characterization and Consequences
(BNE characterization, symmetric BNE, solving, uniqueness, revenue)